GAUSSIAN PROCESSES
EXERCISE SHEET 10: GAUSSIAN FREE FIELD II

Exercise 1 (Green's Function from Continuous-Time Random Walk)**.** *Let $G = (V, E)$ be a finite, connected graph and $V_\partial \in V$ a non-empty boundary. Consider a continuous-time random walk (CTRW) $(X_t)_{t\geq 0}$ on G with unit jump rates at each edge: when the walk is at x, it waits an* $\mathrm{Exp}(d_x)$ *time, then jumps to a uniformly chosen neighbor of x.*

The Green's function $G(v, w)$ of this CTRW is given by

$$G(v, w) \ := \int_0^\tau \mathbb{P}_v(X_t = w)\, dt,$$

where $v, w \in V \setminus V_\partial$, τ is the time that X_t (starting from v) hits the boundary. Show that: G is the covariance matrix of the Gaussian Free Field on the graph. Furthermore, it is symmetric and positive definite.

Exercise 2 (Poisson Equation)**.** *Let $G = (V, E)$ be a finite, connected graph and $V_\partial \in V$ a non-empty boundary. Let $f : V \setminus V_\partial \to \mathbb{R}$ be a given function. Consider the following discrete Poisson equation*

$$\Delta F = f,$$

where $F : V \to \mathbb{R}$, $F|_{V_\partial} = 0$ and Δ denotes the discrete Laplacian operator on the graph. Using the continuous-time random walk in Exercise 1 to construct the solution for this equation..

Exercise 3 (Massive Gaussian Free Field and Killed Random Walks)**.** *Let $G = (V, E)$ be a finite, connected graph, and fix a parameter $m > 0$ (the mass).*

(a) Define the massive Gaussian free field $\phi = (\phi_x)_{x\in V}$ by the density

$$\mathbb{P}(\phi) \propto \exp\left[-\frac{1}{2}\sum_{x\sim y}(\phi_x - \phi_y)^2 - \frac{m^2}{2}\sum_{x\in V}\phi_x^2 \right] \prod_{x\in V} d\phi_x.$$

Show that this defines a centered Gaussian vector $(\phi_x)_{x\in V}$ with covariance matrix

$$\mathrm{Cov}(\phi_x, \phi_y) = \big(\Delta + m^2 I\big)^{-1}(x, y),$$

where Δ is the discrete Laplacian on G.

(b) Let $(X_t)_{t\geq 0}$ be the continuous-time simple random walk on G, and consider an independent exponential killing at rate m^2. Denote by $G^m(x, y)$ the Green's function of the killed random walk:

$$G^m(x, y) := \int_0^\infty \mathbb{P}_x(X_t = y,\ \textit{walk not killed before } t)\, dt.$$

Show that the covariance of the massive GFF is exactly given by this killed Green's function:

$$\mathrm{Cov}(\phi_x, \phi_y) = G^m(x, y).$$